Higher-order correlations for fluctuations in the presence of fields

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The higher-order moments of the fluctuations for thermodynamic systems in the presence of fields are investigated in the framework of a theoretical method. The method uses a generalized statistical ensemble consistent with an adequate expression for the internal energy. The applications refer to the case of a system in a magnetoquasistatic field. In the case of linear magnetic media, one finds that, for the description of the magnetic induction fluctuations, the Gaussian approximation is satisfactory. For nonlinear media, the corresponding fluctuations are non-Gaussian, having a non-null asymmetry. Furthermore, the respective fluctuations have characteristics of leptokurtic, mesokurtic and platykurtic type, depending on the value of the magnetic field strength as compared with a scaling factor of the magnetization curve.

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I. INTRODUCTION

In our previous work [1] we have presented a phenomenological approach to the fluctuations for thermodynamical systems in the presence of electromagnetic fields. In this work the fluctuations were evaluated only by second order numerical characteristics (correlations and moments). From a more general probabilistic perspective [2–6] the fluctuations in physical systems must be evaluated also by means of higher order numerical characteristics (higher order correlations and moments). In the present paper, we present an approach for the evaluation of higher order correlations that characterize the systems considered.

In Sec. II we present the general considerations and relations regarding the problems mentioned. In Sec. III we proceed to evaluate higher order moments for linear magnetic systems (in the presence of magnetoquasistatic field), as well as for nonlinear magnetic media. The results provide relations which might be compared with experimental data.

II. THEORETICAL CONSIDERATIONS

A. Statistical ensembles for generalized systems

Following our previous work [1] we consider a generalized system described by the set of extensive and field variables $(U, X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m)$, where U denotes the generalized internal energy, X_i $(i=1,2,\ldots,n)$ signify the usual extensive variables, excepting the entropy (for systems in the absence of fields), while Y_j $(j=1,2,\ldots,m)$ mean the additional variables that arise due to the presence of the fields.

In the framework of fluctuation theory, the thermodynamical quantities represent the mean (or expected) values of random variables. In the following we denote by $\overline{A}$ the expectation of the random variable A. In this context, we define $U=\overline{E}$, where E denotes the energy regarded as a random variable.

We take the investigated system as a small part of a large, isolated ensemble. Both the system and its environment are macroscopic bodies. In the following the quantities X_i and Y_j are defined as fluctuating (random) variables. The joint probability density function of the quantities E, $\{X_i\}_{i=1}^{n}$ and $\{Y_j\}_{j=1}^{m}$ can be expressed as [7–9]

$$w = Z^{-1} \exp\left(-\beta E - \sum_{i=1}^{n} \alpha_i X_i - \sum_{j=1}^{m} \gamma_j Y_j \right). \tag{1}$$

Here the independent random variables are phase space coordinates (q_r, p_r), X_i and Y_j.

In Eq. (1) the normalization factor is determined by

$$Z = \int \cdots \int \prod_{i=1}^{n} dX_i \exp(-\alpha_i X_i) \prod_{j=1}^{m} dY_j \exp(-\gamma_j Y_j) \times \int_{\Gamma} e^{-\beta E} d\Gamma, \tag{2}$$

where $d\Gamma$ represents the elementary volume in phase space.

The statistical integral Z is a function of the quantities β, $\{\alpha_i\}_{i=1}^{n}$ and $\{\gamma_j\}_{j=1}^{m}$. Hence,

$$d(\ln Z) = \frac{1}{Z}\frac{\partial Z}{\partial \beta} d\beta + \frac{1}{Z}\sum_{i=1}^{n} \frac{\partial Z}{\partial \alpha_i} d\alpha_i + \frac{1}{Z}\sum_{j=1}^{m} \frac{\partial Z}{\partial \gamma_j} d\gamma_j$$

where

$$\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\int E w \, d\Omega = -\overline{E},$$

$$\frac{1}{Z}\frac{\partial Z}{\partial \alpha_i} = -\int X_i w \, d\Omega = -\overline{X_i},$$

$$\frac{1}{Z}\frac{\partial Z}{\partial \gamma_i} = -\int Y_j w \, d\Omega = -\overline{Y_j},$$

$$d\Omega = \prod_{i=1}^{n} dX_i \prod_{j=1}^{m} dY_j d\Gamma.$$

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The physical significance of the parameters β, α_i and γ_j can be identified from

$$d\left(\ln Z+\beta\overline{E}+\sum_{i=1}^{n}\alpha_i\overline{X_i}+\sum_{j=1}^{m}\gamma_j\overline{Y_j}\right)=\beta d\overline{E}+\sum_{i=1}^{n}\alpha_i d\overline{X_i}+\sum_{j=1}^{m}\gamma_j d\overline{Y_j}\tag{3}$$

or using $U=\overline{E}$

$$d\left(\ln Z+\beta U+\sum_{i=1}^{n}\alpha_i\overline{X_i}+\sum_{j=1}^{m}\gamma_j\overline{Y_j}\right)=\beta dU+\sum_{i=1}^{n}\alpha_i d\overline{X_i}+\sum_{j=1}^{m}\gamma_j d\overline{Y_j}.\tag{4}$$

From thermodynamics in the presence of fields [10,11] we have

$$dU=TdS+\sum_{i=1}^{n}\hat{\xi}_i d\overline{X_i}+\sum_{j=1}^{m}\psi_j d\overline{Y_j},\tag{5}$$

where S, T, and $\hat{\xi}_i$ denote respectively nonfield entropy, field dependent temperature, and other intensive parameters, whereas ψ_j denote field variables.

From Eqs. (4) and (5) one obtains

$$d\left(\ln Z+\beta U+\sum_{i=1}^{n}\alpha_i\overline{X_i}+\sum_{j=1}^{m}\gamma_j\overline{Y_j}\right)=\beta TdS+\sum_{i=1}^{n}(\alpha_i+\beta\hat{\xi}_i)d\overline{X_i}+\sum_{j=1}^{m}(\gamma_j+\beta\psi_j)d\overline{Y_j}.\tag{6}$$

The left-hand side of Eq. (6) is an exact differential. The number of terms on the right-hand side depends on the field constraints [10,11]. Furthermore as it must be also an exact differential the following relations must hold,

$$\beta=\frac{1}{kT},\tag{7}$$

$$\alpha_i=-\beta\hat{\xi}_i=-\frac{\hat{\xi}_i}{kT},\tag{8}$$

$$\gamma_j=-\beta\psi_j=-\frac{\psi_j}{kT},\tag{9}$$

where k denotes Boltzmann's constant.

We observe that the quantities $\{\alpha_i\}_{i=1}^{n}$ are functions of the field dependent intensive parameters.

B. Evaluation of higher-order correlations and moments

The mean values of the fluctuating quantities can be evaluated through the following relations:

$$U=\overline{E}=-\left[\frac{\partial(\ln Z)}{\partial\beta}\right]_{\alpha_i,\gamma_j},\quad i=1,2,\ldots,n,\quad j=1,2,\ldots,m\tag{10}$$

$$\overline{X_i}=-\left[\frac{\partial(\ln Z)}{\partial\alpha_i}\right]_{\beta,\alpha_l,\gamma_j},\quad l\neq i,\tag{11}$$

$$\overline{Y_j}=-\left[\frac{\partial(\ln Z)}{\partial\gamma_j}\right]_{\beta,\alpha_i,\gamma_l},\quad l\neq j.\tag{12}$$

Expressions of the type $\overline{\Pi_i(\delta X_i)^{r_i}\Pi_j(\delta Y_j)^{s_j}}$ are called higher-order correlations. By using the statistical sum Z, as introduced above, for some of the respective correlations one obtains the expressions

$$\overline{\delta X_a\delta X_b}=\frac{\partial^2(\ln Z)}{\partial\alpha_a\partial\alpha_b}=-\frac{\partial\overline{X_b}}{\partial\alpha_a}=kT\frac{\partial\overline{X_b}}{\partial\hat{\xi}_a},\quad a,b=1,2,\ldots,n,\tag{13}$$

$$\overline{\delta Y_a\delta Y_b}=\frac{\partial^2(\ln Z)}{\partial\gamma_a\partial\gamma_b}=-\frac{\partial\overline{Y_b}}{\partial\gamma_a}=kT\frac{\partial\overline{Y_b}}{\partial\psi_a},\quad a,b=1,2,\ldots,m,\tag{14}$$

$$\overline{\delta X_a\delta Y_b}=\frac{\partial^2(\ln Z)}{\partial\alpha_a\partial\gamma_b}=-\frac{\partial\overline{Y_b}}{\partial\alpha_a}=kT\frac{\partial\overline{Y_b}}{\partial\hat{\xi}_a},\quad a=1,2,\ldots,n,\quad b=1,2,\ldots,m,\tag{15}$$

$$\overline{\delta X_a\delta X_b\delta X_c}=-\frac{\partial^3(\ln Z)}{\partial\alpha_a\partial\alpha_b\partial\alpha_c}=\frac{\partial^2\overline{X_c}}{\partial\alpha_a\partial\alpha_b}=k^2T^2\frac{\partial^2\overline{X_c}}{\partial\hat{\xi}_a\partial\hat{\xi}_b},\quad a,b,c=1,2,\ldots n,\tag{16}$$

$$\overline{\delta Y_a\delta Y_b\delta Y_c}=-\frac{\partial^3(\ln Z)}{\partial\gamma_a\partial\gamma_b\partial\gamma_c}=\frac{\partial^2\overline{Y_c}}{\partial\gamma_a\partial\gamma_b}=k^2T^2\frac{\partial^2\overline{X_c}}{\partial\psi_a\partial\psi_b},\quad a,b,c=1,2,\ldots m,\tag{17}$$

$$\overline{\delta X_a\delta Y_b\delta Y_c}=-\frac{\partial^3(\ln Z)}{\partial\alpha_a\partial\gamma_b\partial\gamma_c}=\frac{\partial^2\overline{Y_c}}{\partial\alpha_a\partial\gamma_b}=k^2T^2\frac{\partial^2\overline{Y_c}}{\partial\hat{\xi}_a\partial\psi_b},\quad a=1,2,\ldots n,\quad b,c=1,2,\ldots,m,\tag{18}$$

The formulas for the correlations of orders higher than 3 are generally more complicated. The higher-order moments can be obtained by means of the following recurrence formulas [7]:

$$\overline{(\delta X_a)^{n+1}}=-\frac{\partial}{\partial\alpha_a}\overline{(\delta X_a)^n}-n\overline{(\delta X_a)^{n-1}}\frac{\partial\overline{X_a}}{\partial\alpha_a},\tag{19}$$

$$\overline{(\delta Y_a)^{n+1}} = -\frac{\partial}{\partial \gamma_a}\overline{(\delta Y_a)^n} - n\overline{(\delta Y_a)^{n-1}}\frac{\partial \overline{Y_a}}{\partial \gamma_a}. \quad (20)$$

As examples we give here the expressions of the moments $\overline{(\delta X_a)^4}$ and $\overline{(\delta Y_a)^4}$:

$$\overline{(\delta X_a)^4} = -\frac{\partial^3 \overline{X_a}}{\partial \alpha_a^3} + 3\left(\frac{\partial \overline{X_a}}{\partial \alpha_a}\right)^2 = \left(kT\frac{\partial}{\partial \hat{\xi}_a}\right)^3 \overline{X_a} + 3\left(kT\frac{\partial \overline{X_a}}{\partial \hat{\xi}_a}\right)^2, \quad (21)$$

$$\overline{(\delta Y_a)^4} = -\frac{\partial^3 \overline{Y_a}}{\partial \gamma_a^3} + 3\left(\frac{\partial \overline{Y_a}}{\partial \gamma_a}\right)^2 = \left(kT\frac{\partial}{\partial \psi_a}\right)^3 \overline{Y_a} + 3\left(kT\frac{\partial \overline{Y_a}}{\partial \psi_a}\right)^2. \quad (22)$$

The fourth order moments are of interest for the evaluation of the so-called excess coefficient,

$$C_E = \frac{\overline{(\delta X_a)^4}}{[\overline{(\delta X_a)^2}]^2} - 3, \quad (23)$$

which is an indicator of the deviation from the Gaussian distribution [2].

III. HIGHER-ORDER MOMENTS FOR SYSTEMS IN A MAGNETOQUASISTATIC FIELD

A. Linear magnetic media

Let us consider a uniformly magnetized continuous medium, situated in a magnetoquasistatic field. The system is characterized by the extensive parameters $(U,V,N,\mathbf{B})$, where V, N and $\mathbf{B}$ denote the volume, number of particles and magnetic induction respectively. In the case of linear magnetic systems the differential of the internal energy is given by [10,11]

$$dU = TdS - \hat{p}dV + \hat{\zeta}dN + V\mathbf{H}\cdot d\mathbf{B}. \quad (24)$$

For the sake of brevity we omitted the mean symbol from above the parameters V, N and $\mathbf{B}$, which will continue to represent mean values henceforth.

Equations (13) and (16) show that the moments associated with the usual thermodynamic quantities, i.e., volume V or of particles number N, are functions of the parameters $\hat{\xi}_i$, which depend on the field constraints.

For example, in the case $\mathbf{B}=$const one obtains for the volume V

$$\overline{(\delta V)^2} = -kT\left(\frac{\partial V}{\partial \hat{p}}\right)_{T,\hat{\zeta},\mathbf{B}}, \quad (25)$$

$$\overline{(\delta V)^3} = k^2T^2\left(\frac{\partial^2 V}{\partial \hat{p}^2}\right)_{T,\hat{\zeta},\mathbf{B}}, \quad (26)$$

where [10]

$$\hat{p}(\mathbf{B}=\text{const}) = p_{\mathbf{B},N} = p - \frac{1}{2}\mathbf{H}\cdot\mathbf{B} - \frac{1}{2}H^2\rho\frac{\partial \mu}{\partial \rho}, \quad (27)$$

$\mathbf{H}$ signifies the magnetic field strength, μ is the magnetic permeability, and $\rho = N/V$.

By using the properties of Jacobians, Eq. (25) can be transformed as follows:

$$\overline{(\delta V)^2} = -kT\frac{\partial(V,\hat{\zeta},T,\mathbf{B})}{\partial(\hat{p},\hat{\zeta},T,\mathbf{B})} = -kT\frac{\partial(V,\hat{\zeta},T,\mathbf{B})}{\partial(V,N,T,\mathbf{B})}\frac{\partial(V,N,T,\mathbf{B})}{\partial(\hat{p},\hat{\zeta},T,\mathbf{B})} = -kT\frac{\left(\frac{\partial \hat{\zeta}}{\partial N}\right)_{T,V,\mathbf{B}}}{\left(\frac{\partial \hat{p}}{\partial V}\right)_{T,N,\mathbf{B}}\left(\frac{\partial \hat{\zeta}}{\partial N}\right)_{T,V,\mathbf{B}} + \left(\frac{\partial \hat{\zeta}}{\partial V}\right)^2_{T,N,\mathbf{B}}}. \quad (28)$$

In the above relation use was made of $-(\partial\hat{p}/\partial N)_{T,V,\mathbf{B}} = (\partial\hat{\zeta}/\partial V)_{T,N,\mathbf{B}}$. Note that the result (28) is the same as the one obtained [1] within the Gaussian approximation.

We proceed to evaluate the second and third order parameters of fluctuations for the magnetic induction $\mathbf{B}$. For simplicity we suppose that volume and particle number are fixed. Using relations (14), (17) and (22) we find

$$\overline{(\delta B)^2} = \frac{kT}{V}\left(\frac{\partial B}{\partial H}\right)_{T,V,N} = \frac{kT\mu}{V}, \quad (29)$$

$$\overline{(\delta B)^3} = \left(\frac{kT}{V}\right)^2\left(\frac{\partial^2 B}{\partial H^2}\right)_{T,V,N} = \left(\frac{kT}{V}\right)^2\frac{\partial \mu}{\partial H} = 0. \quad (30)$$

Equation (29) is identical to the one obtained [1] within the Gaussian approximation. Note that, in the case of linear magnetic media, $\overline{(\delta B)^3}$ vanishes, because μ is independent of $\mathbf{H}$. In this case the excess coefficient (23) also vanishes. These facts show that, in the case alluded to, the Gaussian approximation is sufficient for a quantitative description of the fluctuations of B.

B. Nonlinear magnetic media

In the case of nonlinear magnetic media, μ depends on $\mathbf{H}$. Therefore, evaluation of the moments of orders higher than 2 becomes necessary.

We approach this case under the constraints $V=$const and $N=$const, so that the internal energy U reduces to

$$dU = TdS + V\mathbf{H}\cdot d\mathbf{B}.$$

This gives for the moments of 2, 3 and 4 order of $\mathbf{B}$:

$$\overline{(\delta B)^2} = \frac{kT}{V}\left(\frac{\partial B}{\partial H}\right)_{T,V,N}, \quad (31)$$

$$\overline{(\delta B)^3} = \left(\frac{kT}{V}\right)^2 \left(\frac{\partial^2 B}{\partial H^2}\right)_{T,V,N}, \tag{32}$$

$$\overline{(\delta B)^4} = \left(\frac{kT}{V}\right)^3 \left(\frac{\partial^3 B}{\partial H^3}\right)_{T,V,N} + 3\left[\frac{kT}{V}\left(\frac{\partial B}{\partial H}\right)_{T,V,N}\right]^2$$

$$= \left(\frac{kT}{V}\right)^3 \left(\frac{\partial^3 B}{\partial H^3}\right)_{T,V,N} + 3[\overline{(\delta B)^2}]^2. \tag{33}$$

In order to find the explicit expressions of $\overline{(\delta B)^2}$, $\overline{(\delta B)^3}$ and $\overline{(\delta B)^4}$ it is necessary to know the expression of the function $B = B(H)$. To this end we use the well known Langevin equation:

$$B = \mu_0 M_s \left(\coth a - \frac{1}{a}\right) + \mu_0 H, \tag{34}$$

where

$$a = \frac{\mu_0 m H}{kT}, \tag{35}$$

M_s represents the saturation magnetization, μ_0 is the vacuum permeability, and m signifies the magnetic moment of an individual molecule.

By means of some simple mathematical operations one finds

$$\overline{(\delta B)^2} = \frac{kT\mu_0}{V}\left[\frac{\mu_0 m M_s}{kT}\left(\frac{1}{a^2} - \frac{1}{\sinh^2 a}\right) + 1\right], \tag{36}$$

$$\overline{(\delta B)^3} = \frac{2\mu_0^3 m^2 M_s}{V^2}\left(\frac{\cosh a}{\sinh^3 a} - \frac{1}{a^3}\right), \tag{37}$$

$$\overline{(\delta B)^4} = \frac{2\mu_0^4 m^3 M_s}{V^3}\left(\frac{3}{a^4} + \frac{\sinh^2 a - 3\cosh^2 a}{\sinh^4 a}\right) + 3\left\{\frac{kT\mu_0}{V}\left[\frac{\mu_0 m M_s}{kT}\left(\frac{1}{a^2} - \frac{1}{\sinh^2 a}\right) + 1\right]\right\}^2. \tag{38}$$

Another functional dependence of B on H is given [12] by

$$B = \mu_0 M_s \left[1 - \exp\left(-\frac{H^2}{2\sigma^2}\right)\right] + \mu_0 H, \tag{39}$$

where σ is a scaling factor.

In this case the differential permeability $\mu_d = dB/dH$ is given by the formula

$$\mu_d = \mu_0\left(1 + \frac{M_s H}{\sigma^2} e^{-H^2/2\sigma^2}\right).$$

From this relation it follows that

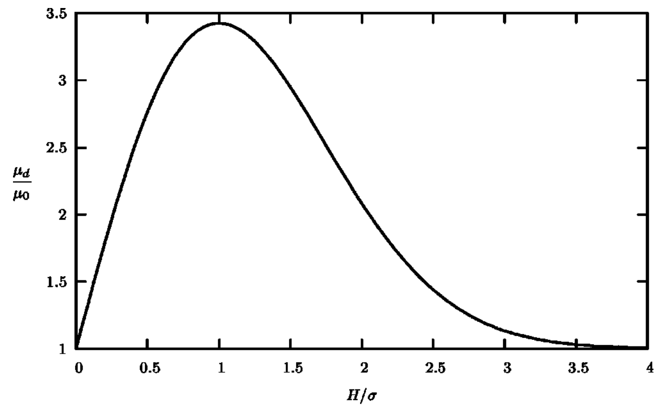

FIG. 1. A plot of μ_d/μ_0 for the case $M_s = 4\sigma$.

$$\frac{d\mu_d}{dH} = \frac{\mu_0 M_s}{\sigma^2}\left(1 - \frac{H^2}{\sigma^2}\right) e^{-H^2/2\sigma^2}$$

and

$$\frac{d^2\mu_d}{dH^2} = -\frac{\mu_0 M_s H}{\sigma^4}\left(3 - \frac{H^2}{\sigma^2}\right) e^{-H^2/2\sigma^2}.$$

For $H = \sigma$ we have $d\mu_d/dH = 0$ and $d^2\mu_d/dH^2 < 0$. These conditions imply a maximum for the differential permeability at $H = \sigma$.

Figure 1 shows a plot of μ_d/μ_0 for the case $M_s = 4\sigma$.

By using the general formulas for the 2nd, 3rd, and 4th order moments of the random variable B one obtains

$$\overline{(\delta B)^2} = \frac{\mu_0 kT}{V}\left\{\frac{M_s H}{\sigma^2}\exp\left(-\frac{H^2}{2\sigma^2}\right) + 1\right\}, \tag{40}$$

$$\overline{(\delta B)^3} = \left(\frac{kT}{V}\right)^2 \frac{\mu_0 M_s}{\sigma^2}\left(1 - \frac{H^2}{\sigma^2}\right)\exp\left(-\frac{H^2}{2\sigma^2}\right), \tag{41}$$

$$\overline{(\delta B)^4} = \left(\frac{kT}{V}\right)^3 \frac{\mu_0 M_s H}{\sigma^4}\left(\frac{H^2}{\sigma^2} - 3\right)\exp\left(-\frac{H^2}{2\sigma^2}\right) + 3\left(\frac{\mu_0 kT}{V}\right)^2\left\{\frac{M_s H}{\sigma^2}\exp\left(-\frac{H^2}{2\sigma^2}\right) + 1\right\}^2. \tag{42}$$

Finally we wish to note the following observations.

(1) $\overline{(\delta B)^3}$ change its sign at the point $H = \sigma$, where the differential permeability $\mu_d = dB/dH$ takes its maximal value [this means the inflection point of the function $B = B(H)$]. For $H < \sigma$, the moment $\overline{(\delta B)^3}$ is positive while for $H > \sigma$ it is negative. Figure 2 shows a plot of the function $f_3 = \overline{(\delta B)^3}/b$, where $b = (kT/V)^2\mu_0/\sigma$.

(2) The fourth order moment $\overline{(\delta B)^4}$ gives information about the deviation from the Gaussian approximation. This is because it is implied in the so-called excess coefficient (23). In the case discussed here we have

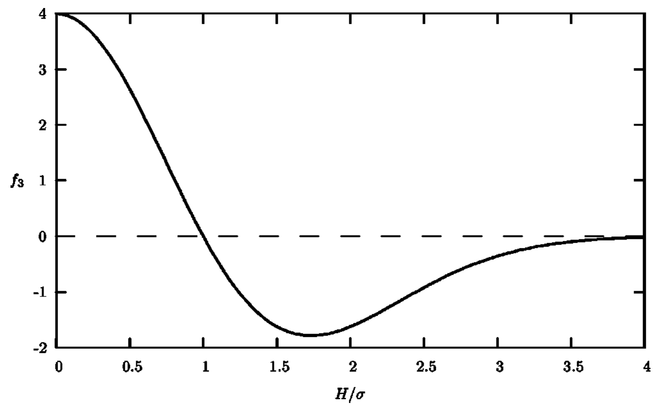

FIG. 2. A plot of the function f_3 for the case $M_s = 4\sigma$.

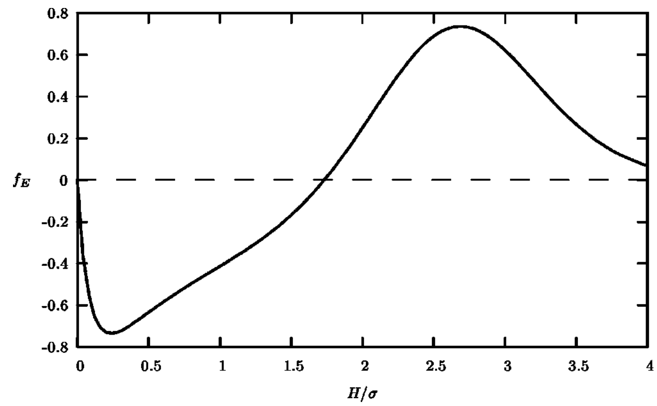

FIG. 3. A plot of the function f_E for the case $M_s = 4\sigma$.

$$C_E = \frac{kT}{V}\,\frac{M_s H}{\mu_0 \sigma^4}\,\frac{\left(\dfrac{H^2}{\sigma^2}-3\right)\exp\left(-\dfrac{H^2}{2\sigma^2}\right)}{\left\{\dfrac{M_s H}{\sigma^2}\exp\left(-\dfrac{H^2}{2\sigma^2}\right)+1\right\}^2}. \qquad (43)$$

In probabilistic terminology [13] the distribution of a random variable is called leptokurtic, mesokurtic or platykurtic as the excess coefficient C_E satisfies the conditions $C_E > 0$, $C_E = 0$ and $C_E < 0$ respectively. Then, by taking into account the expression (43) of C_E one can say that, for the situation studied here the fluctuations of the magnetic induction B are leptokurtic, mesokurtic and platykurtic as the magnetic field strength H satisfies the conditions $H > \sqrt{3}\sigma$, $H = \sqrt{3}\sigma$, and $H < \sqrt{3}\sigma$, respectively. Figure 3 shows a plot of the function

$$f_E = C_E \frac{V \mu_0 \sigma^2}{kT}.$$

IV. SUMMARY AND CONCLUSIONS

(1) We investigated the higher-order moments of the fluctuations for complex thermodynamic systems (i.e., systems considered in the presence of fields). Our approach uses a generalized statistical ensemble. We considered the case when the energy E, the usual extensive parameters $\{X_i\}_{i=1}^{n}$ and the field parameters $\{Y_j\}_{j=1}^{m}$ are fluctuating random variables. We find that the higher-order moments of fluctuations depend on the field constraints.

(2) The general results from Sec. II were particularized for the case of a system situated in a magnetoquasistatic field. If these systems are magnetically linear, then the third order moment of the magnetic induction and the excess coefficient vanish. Therefore, the description of fluctuations of the magnetic induction can be done in the framework of the Gaussian approximation.

(3) For nonlinear magnetic media $\overline{(\delta B)^3} \neq 0$. Consequently the fluctuations of B deviate from the normal distribution. The respective deviations are characterized by the various values of the excess coefficient C_E given by the formula (43). This formula shows that the fluctuations of B can be leptokurtic, mesokurtic and platykurtic for the cases $H > \sqrt{3}\sigma$, $H = \sqrt{3}\sigma$ and $H < \sqrt{3}\sigma$, respectively.

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